ON THE CHARACTERIZATION OF EXPANSION MAPS FOR SELF-AFFINE TILINGS

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ABSTRACT. We consider self-affine tilings in $\mathbb{R}^n$ with expansion matrix ϕ and address the question which matrices ϕ can arise this way. In one dimension, λ is an expansion factor of a self-affine tiling if and only if $|\lambda|$ is a Perron number, by a result of Lind. In two dimensions, when ϕ is a similarity, we can speak of a complex expansion factor, and there is an analogous necessary condition, due to Thurston: if a complex λ is an expansion factor of a self-similar tiling, then it is a complex Perron number. We establish a necessary condition for ϕ to be an expansion matrix for any n, assuming only that ϕ is diagonalizable over $\mathbb{C}$. We conjecture that this condition on ϕ is also sufficient for the existence of a self-affine tiling.

1. INTRODUCTION

Self-affine tilings arise in many different contexts, notably in dynamics (Markov partitions for hyperbolic maps [21, 11, 16]), logic (aperiodic tilings [15]), number theory (radix representations [19, 13]), physics (quasicrystals [3]), ergodic theory [22], and hyperbolic groups [4]. See [2, 20] for recent surveys with a large bibliography.

A **self-affine tiling** (SAT) $\mathcal{T} = \{T_i\}_{i\in I}$ of $\mathbb{R}^n$ is a covering of $\mathbb{R}^n$ with sets (tiles) T_i satisfying the following properties:

(1) Each tile T_i is the closure of its interior.
(2) Interiors of tiles do not overlap.
(3) There are a finite number of tile types up to translation.
(4) The tiling is **repetitive** and has **finitely many local configurations** (see the next section for definitions).

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(5) There is an expanding linear map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ mapping tiles over tiles: the image of a tile T_i is a union of tiles of $\mathcal{T}$, and two tiles of the same type have images which are translation-equivalent patches of tiles.

The simplest example is the periodic tiling with unit cubes and expansion mapping $\phi(x) = 2x$. However typically SATs are nonperiodic and have tiles with fractal boundaries. See Figures 1 and 2 for examples in $\mathbb{R}^2$.

Lind [14] (using different language) gives a characterization of expansion factors of self-affine tilings in one dimension: λ is the expansion of an SAT of $\mathbb{R}$ if and only if $|\lambda|$ is a **Perron number**, that is, a real algebraic integer which is strictly larger in modulus than all of its Galois conjugates.

A self-affine tiling is **self-similar** if ϕ is a similarity (a homothety followed by a rotation). Thurston [24] showed that the expansion factor $\lambda \in \mathbb{C}$ of a self-similar tiling of $\mathbb{R}^2$ is a **complex Perron number**, that is, an algebraic integer which is strictly larger in modulus than its Galois conjugates except for its complex conjugate. In [9], a construction of a self-similar tiling for every complex Perron number is given; unfortunately, the proof as written in subsection 4.5 of [9] is incomplete. A version of the construction does yield a tiling with expansion λ^k for k sufficiently large, and we hope that it can be modified to get a tiling with expansion λ, completing the characterization. This gap does not affect the construction in section 6 of [9] which uses free group endomorphisms; however, the latter does not cover all the complex Perron numbers. See also [5] for a related construction.

In the current paper we study SATs of $\mathbb{R}^n$ with expansion matrix ϕ which is diagonalizable over $\mathbb{C}$. We show that if ϕ is the expansion matrix for an SAT then eigenvalues of ϕ are algebraic integers, and for every eigenvalue γ, all Galois conjugates of γ which have modulus $\geq |\gamma|$ have multiplicity (among eigenvalues of ϕ) at least as large as that of γ, see Theorem 3.1 below.

An alternative description of this criterion is that there is an integer matrix M acting on $\mathbb{R}^N$ for some $N \geq n$, which has an invariant real subspace W of dimension n, on which it has strictly larger growth (that is, strictly larger determinant, in absolute value) than for any other n-dimensional invariant subspace, and M restricted to W is linearly conjugate to ϕ.

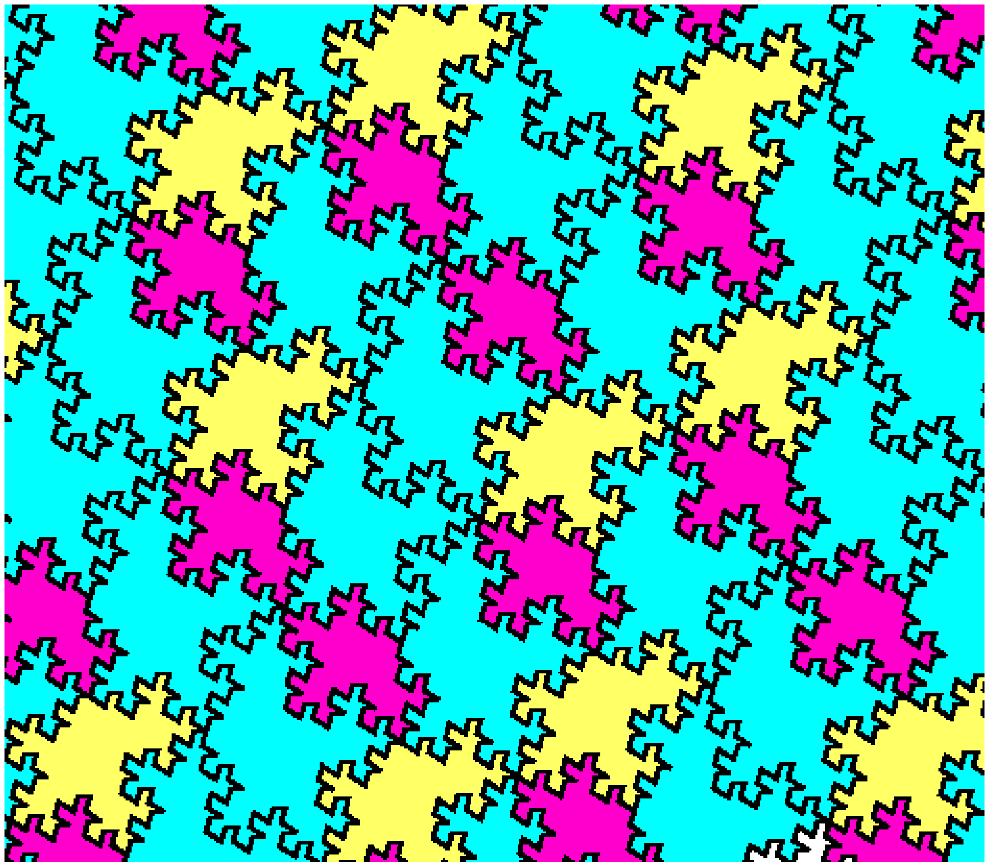

FIGURE 1. A self-affine tiling in the plane with expansion $\phi(z) = \lambda z$ where λ is the complex root of $x^3 + x + 1 = 0$. Here there are three tile types, all similar. The smallest scales to the medium which scales to the large; the large subdivides into a small and a large. One can construct this tiling using the method of [9, Sec.6], as follows. To a reduced word in the free group on three letters $F(a, b, c)$ associate a polygonal path in $\mathbb{C}$ by sending $a^{\pm 1}$ to ± 1, $b^{\pm 1}$ to $\pm\lambda$, $c^{\pm 1}$ to $\pm\lambda^2$. Let ψ be the endomorphism of $F(a, b, c)$ defined by $\psi(a) = b, \psi(b) = c, \psi(c) = a^{-1}b^{-1}$. Consider the three commutators $[a, b] = aba^{-1}b^{-1}, [b, c]$, and $[a, c]$; they represent three closed paths. Then $\lim_{n\to\infty} \lambda^{-n}\psi^n([a, c])$ is the boundary of the smallest tile; the other tiles boundaries are $\lim_{n\to\infty} \lambda^{-n}\psi^n([a, b])$ and $\lim_{n\to\infty} \lambda^{-n}\psi^n([b, c])$. The subdivision rule comes from the identities $\psi([a, c]) = a^{-1}[a, b]a$, $\psi[a, b] = [b, c]$ and $\psi[b, c] = [c, a^{-1}b^{-1}] = (a^{-1}[a, c]a)(a^{-1}b^{-1}[b, c]ba)$.

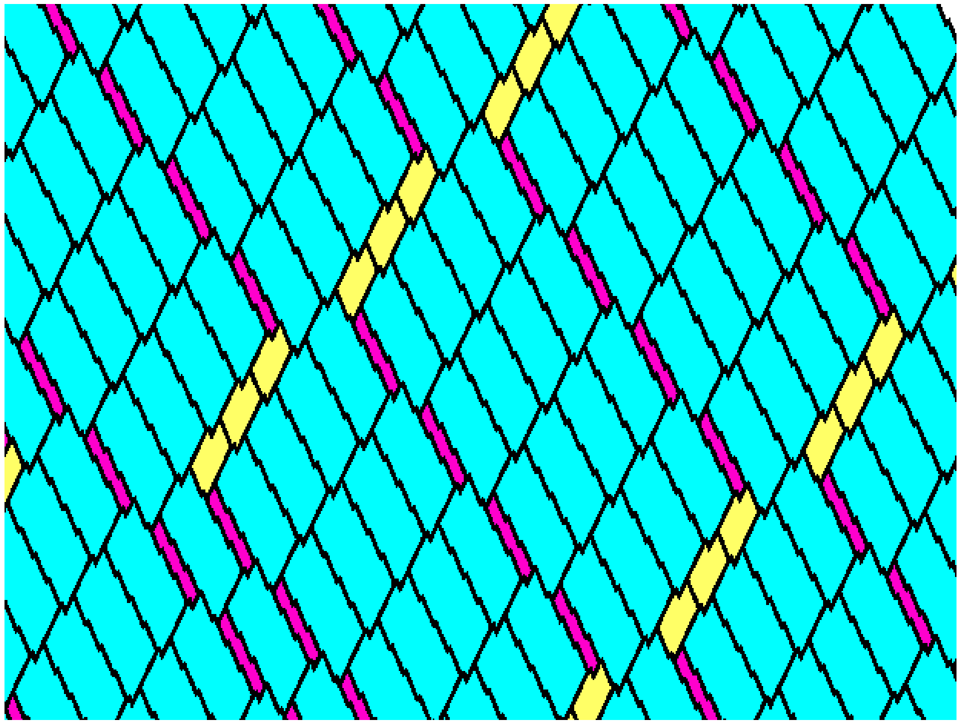

FIGURE 2. A self-affine tiling in the plane with diagonal expansion matrix $\mathrm{Diag}[x_1, x_2]$ where $x_1 \approx 2.19869,\ x_2 \approx -1.91223$ are roots of $x^3 - x^2 - 4x + 3 = 0$.

The converse to our result is open: does there exist, for every linear map ϕ satisfying the above conditions, an SAT with expansion ϕ? We conjecture that the answer is yes.

In Figure 2 we show an example of a self-affine (non-self-similar) SAT in the plane. The subdivision rule is indicated in Figure 3.

Our methods do not at present extend to the non-diagonalizable case. However, we conjecture that the second description above holds in general, that is, without the constraint of diagonalizability, ϕ is the expansion of an SAT if and only if there is an integer matrix M acting on $\mathbb{R}^N$ for some $N \geq n$, which has an invariant real subspace W of dimension n, on which it has strictly larger growth (determinant) than for any other n-dimensional invariant subspace, and M restricted to W is linearly conjugate to ϕ. For example, we conjecture that there is no SAT in $\mathbb{R}^3$

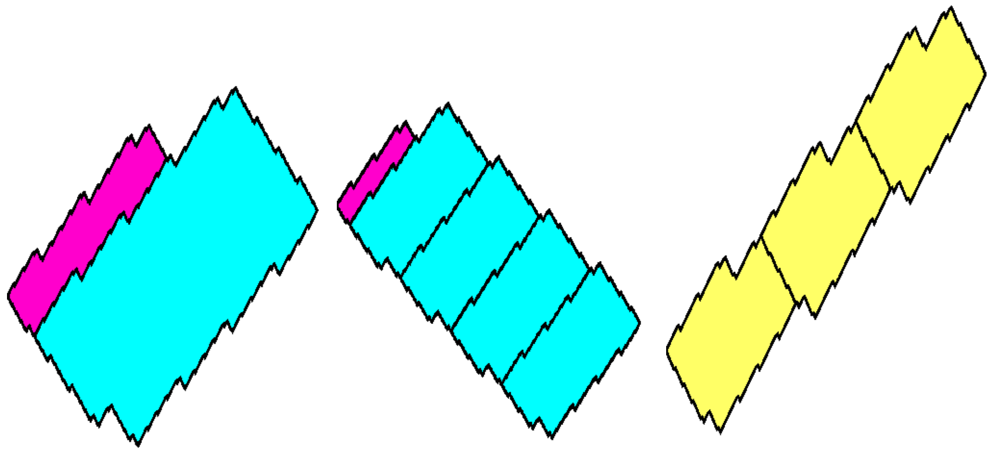

FIGURE 3. Subdivision rule: $1 \to \{3,2\}$, $2 \to \{3,2,2,2,2\}$, $3 \to \{1,1,1\}$. The construction is similar to the previous example but with a, b, c corresponding to vectors $(1,1), (x_1 - 1, x_2 - 1), (x_1^2 - x_1, x_2^2 - x_2)$ in $\mathbb{R}^2$, endomorphsm $\psi(a) = ab, \psi(b) = c, \psi(c) = ab^4$ and tiles $[b,a], [b,c], [a,c]$.

with expansion

$$\begin{pmatrix} 3+\sqrt{2} & 1 & 0 \\ 0 & 3+\sqrt{2} & 0 \\ 0 & 0 & 3-\sqrt{2} \end{pmatrix}$$

although it is easy to construct one with expansion

$$\begin{pmatrix} 3+\sqrt{2} & 0 & 0 \\ 0 & 3+\sqrt{2} & 0 \\ 0 & 0 & 3-\sqrt{2} \end{pmatrix}$$

2. PRELIMINARIES

We say that a tiling $\mathcal{T} = \{T_i\}_{i \in I}$ has a finite number of tile types up to translation, if there is an equivalence relation $\sim$ on the tiles T_i with a finite number of equivalence classes and $T_i \sim T_j$ implies that T_j is a translate of T_i. We denote $[T_i]$ the equivalence class of tile T_i, and say T_i is a tile of **type** $[T_i]$.

A **patch** in a tiling is a finite set of its tiles. Two patches are said to be **equivalent** if one is a translate of the other, that is, there is a single translation which takes every tile in one patch to an equivalent tile in the other patch. The **radius of a patch** is the radius of the smallest ball containing the patch.

A tiling is said to have a **finite number of local configurations**, or FLC for short, if there are a finite number of equivalence classes of patches, up to translation, of any given radius.

An FLC tiling is **repetitive** if for all $r > 0$ there is an $R > 0$ such that every patch of radius r can be found, up to translation, in any ball of radius R in the tiling. This is equivalent to minimality of the orbit closure of the tiling, see e.g. [18], and was called quasiperiodicity in [24, 10].

In an SAT, the ϕ-image of each tile type is a well-defined collection of translates of tile types. If T_i is a tile we can write $\phi T_i = \cup_j (T_{i_j} + d_{i_j})$, which is a finite interior-disjoint union. This subdivision only depends on the type of tile T_i, in the sense that equivalent tiles have equivalent subdivisions. In particular we let m_{ij} be the number of tiles of type j in the subdivision of a tile of type i. The matrix $\mathbf{m} = (m_{ij})$ is the **subdivision matrix**, it is a nonegative integer matrix which is **primitive**: some power is strictly positive (by repetitivity of the tiling). The leading eigenvalue of $\mathbf{m}$ is the volume expansion of the SAT, which therefore must be a real Perron number.

Given an SAT, one can select in each of the tile types a point, called a **control point**, in such a way that the set $\mathcal{C}$ of the control points of tiles in a tiling is forward invariant under ϕ: $\phi\mathcal{C} \subset \mathcal{C}$. This can be accomplished as follows [24] (see also [16, Prop. 1.3]): for each tile type $[T_i]$, select one tile in its image under expansion and subdivision. Let the preimage of this tile be $A[T_i] \subset [T_i]$. Then the sequence $[T_i], A[T_i], A(A[T_i]), \ldots$ nests down to a single point in $[T_i]$, denoted by $c(T_i)$, which we define to be the control point of T_i. For a tile $T = T_i + x$ we let $c(T) = c(T_i) + x$.

3. Theorem

The following theorem is stated in [10].

Theorem 3.1. *Let ϕ be a diagonalizable (over $\mathbb{C}$) expanding linear map on $\mathbb{R}^n$, and let $\mathcal{T}$ be a self-affine tiling of $\mathbb{R}^n$ with expansion ϕ. Then*

(i) *every eigenvalue of ϕ is an algebraic integer;*

(ii) *if λ is an eigenvalue of ϕ of multiplicity k and γ is an algebraic conjugate of λ, then either $|\gamma| < |\lambda|$, or γ is also an eigenvalue of ϕ of multiplicity greater than or equal to k.*

The proof is based on the arguments of Thurston [24] and Kenyon [10], but we fill several gaps in those arguments and provide a great deal more detail. In particular, Lemmas 3.7 and 3.8 have no analogs in [24, 10]. It should be pointed out that the corresponding parts of [24] and [10] have never appeared in refereed publications, but have been widely cited and used in the literature on tilings and tiling dynamical systems.

By appropriate choice of a basis, we can assume that the linear map ϕ has the real canonical form, see [7, Th. 6.4.2]. Since ϕ is diagonalizable over $\mathbb{C}$, this means that we have a direct sum decomposition

$$\mathbb{R}^n = \bigoplus_{i=1}^{p} E_i \tag{1}$$

into invariant subspaces associated with eigenvalues λ_i of ϕ, where we count eigenvalues, having non-negative imaginary part, with multiplicities. For a real eigenvalue λ_i, the subspace E_i is one-dimensional, and $\phi|_{E_i}$ acts as multiplication by λ_i. For a non-real eigenvalue λ_i, the subspace E_i is two-dimensional. Identifying it with a complex plane, we get that $\phi|_{E_i}$ acts as multiplication by the complex number λ_i, in other words, as a composition of a dilation and a rotation. We can define a norm on $\|\cdot\|$ on $\mathbb{R}^n$ such that

$$\|x\| = \max_i \|x_i\| \quad \text{for } x = \sum_{i=1}^{p} x_i,\ x_i \in E_i, \quad \|\phi x_i\| = |\lambda_i|\|x_i\| \tag{2}$$

(here $\|x_i\|$ is just the Euclidean norm on E_i in our basis).

Beginning of the proof. Let $\mathcal{C} = \mathcal{C}(\mathcal{T})$ be a set of control points of the tiling $\mathcal{T}$. Recall that $\phi(\mathcal{C}) \subset \mathcal{C}$ by construction. Consider $J = \langle \mathcal{C} \rangle$, the free Abelian group generated by $\mathcal{C}$. It is easy to see that J is finitely generated. Indeed, let

$$\Psi := \{c(T') - c(T) :\ T, T' \in \mathcal{T},\ T \neq T',\ T \cap T' \neq \emptyset\}. \tag{3}$$

The set Ψ is finite by FLC, and J is generated by Ψ and an arbitrary control point (we can get from it to any control point by moving "from neighbor to neighbor"). Let us fix free generators $v_1, \ldots, v_N$ of J. These are vectors in $\mathbb{R}^n$; of course, they need not be in $\mathcal{C}$. They span $\mathbb{R}^n$, since $\mathcal{C}$ is relatively dense. Note that the choice of the generators is non-unique; in fact, we will need to choose them in a specific way at the end of the proof. However, for now any generators will do. Let V be

the matrix $V = [v_1 \ldots v_N]$. This is a $n \times N$ matrix of rank n. By the definition of free generators, for every $\xi \in J$ there exists a unique $a(\xi) \in \mathbb{Z}^N$ such that

$$\xi = Va(\xi). \tag{4}$$

We call $\xi \mapsto a(\xi)$ the "address map." Observe that

$$\mathrm{Span}_{\mathbb{R}}\{a(\xi) : \ \xi \in \mathcal{C}\} = \mathbb{R}^N. \tag{5}$$

Indeed, J is generated by $\mathcal{C}$, hence every v_j is an integral linear combination of control points, and $a(v_j)$ is the jth unit vector in $\mathbb{R}^N$.

Lemma 3.2. *The address map is uniformly Lipschitz on $\mathcal{C}$: there exists $L_1 > 0$ such that*

$$\|a(\xi) - a(\xi')\| \le L_1 \|\xi - \xi'\| \quad \textit{for all } \xi, \xi' \in \mathcal{C}. \tag{6}$$

This lemma is a special case of the implication (i) $\Rightarrow$ (v) in [12, Th. 2.2]. Note that the address map is usually not even continuous on J, since J is not discrete in $\mathbb{R}^n$ unless we have a "lattice tiling," whereas the range of the address map is a subset of the integer lattice in $\mathbb{R}^N$.

Observe that $\phi\mathcal{C} \subset \mathcal{C}$ implies $\phi J \subset J$, hence there exists an integer $N \times N$ matrix M such that

$$\phi V = VM. \tag{7}$$

In other words, we have the commutative diagram (where i indicates the natural inclusion)

$$\begin{tikzcd}
\mathbb{Z}^N \arrow[r, "i"] & \mathbb{R}^N \arrow[r, "M"] \arrow[d, "V"'] & \mathbb{R}^N \arrow[d, "V"] & \mathbb{Z}^N \arrow[l, "i"'] \\
J \arrow[u, "a"'] \arrow[r, "i"] & \mathbb{R}^n \arrow[r, "\phi"] & \mathbb{R}^n & J \arrow[l, "i"'] \arrow[u, "a"]
\end{tikzcd}$$

For every (complex) eigenvalue λ of ϕ we can find a (complex) left eigenvector e_λ of ϕ corresponding to λ. Then $e_\lambda V$ is a left eigenvector for M corresponding to λ (note that $e_\lambda V \ne 0$ since V has maximal possible rank n). This proves (i): every eigenvalue of ϕ is also an eigenvalue of M, hence an algebraic integer. Note also that (7) implies

$$a(\phi\xi) = Ma(\xi), \quad \forall\, \xi \in J. \tag{8}$$

Lemma 3.3. *The matrix M is diagonalizable over $\mathbb{C}$.*

Proof. Recall that J is a free $\mathbb{Z}$-module, on which ϕ acts as an endomorphism, and M is the matrix if this endomorphism in the basis $\mathcal{V} := \{v_1, \ldots, v_N\}$. Note that $\mathbb{Q} \cdot J$ is a vector space over $\mathbb{Q}$, and $\mathcal{V}$ is also a basis of this vector space. Then ϕ induces a linear transformation of $\mathbb{Q} \cdot J$, whose matrix in the basis $\mathcal{V}$ is also M.

Consider the decomposition (1) of $\mathbb{R}^n$ into real eigenspaces E_i corresponding to the eigenvalues λ_i of ϕ. Decomposing the vectors v_j (the generators of J) in terms of E_i yields

$$J \subset J' := \bigoplus_{i=1}^{p} J_i e_i,$$

where $e_i \in E_i$ and J_i is a finitely-generated $\mathbb{Z}[\lambda_i]$-module. (Here we identify two-dimensional subspaces E_i with a complex plane on which ϕ acts as multiplication by λ_i.) Then $\mathbb{Q} \cdot J_i$ is a vector space over $\mathbb{Q}$ and over $\mathbb{Q}(\lambda_i)$ (a field). Let $\{y_1^{(i)}, \ldots, y_{r_i}^{(i)}\}$ be a basis of $\mathbb{Q} \cdot J_i$ over $\mathbb{Q}(\lambda_i)$. Let n_i be the degree of the algebraic integer λ_i. Then $\{\lambda_i^s y_k^{(i)} :\ 0 \le s \le n_i - 1,\ 1 \le k \le r_i,\ i \le p\}$ is a basis for the vector space $\mathbb{Q} \cdot J'$ over $\mathbb{Q}$. In this basis, the linear transformation induced by ϕ has a block matrix, whose every block is a companion matrix of the minimal polynomial of one of the λ_i's. This matrix is diagonalizable over $\mathbb{C}$, since the minimal polynomial has no repeated roots. Finally, we note that the linear transformation induced by ϕ on $\mathbb{Q} \cdot J$ is a restriction of the one which is induced on $\mathbb{Q} \cdot J'$, hence its matrix, M, is diagonalizable as well. □

Now suppose that γ is a conjugate of λ, $\gamma \neq \lambda, \overline{\lambda}$, and $|\gamma| > 1$. Then γ is an eigenvalue of M. Let U_γ be the (real) eigenspace for M corresponding to γ. By Lemma 3.3, there is a projection π_γ from $\mathbb{R}^N$ to U_γ commuting with M. By definition, the only eigenvalues of $M|_{U_\gamma}$ are γ and $\overline{\gamma}$ (if γ is nonreal). Thus, we can fix a norm on U_γ satisfying

$$\|My\| = |\gamma|\, \|y\|, \quad y \in U_\gamma. \tag{9}$$

Consider the mapping $f_\gamma : \mathcal{C} \to U_\gamma$ given by

$$f_\gamma(\xi) = \pi_\gamma a(\xi), \quad \xi \in \mathcal{C}. \tag{10}$$

We would like to extend f_γ to the entire space $\mathbb{R}^n$. We let

$$f_\gamma(\phi^{-k}\xi) = M^{-k} f_\gamma(\xi), \quad \xi \in \mathcal{C}. \tag{11}$$

This is well-defined since M is invertible on U_γ, and unambiguous by (8), since $\pi_\gamma M = M\pi_\gamma$. This way we have f_γ defined on a dense set

$$\mathcal{C}_\infty := \bigcup_{k=0}^{\infty} \phi^{-k}\mathcal{C}.$$

Our goal is to show that f_γ is uniformly continuous on $\mathcal{C}_\infty$, hence can be extended to all of $\mathbb{R}^n$. In fact, it is Hölder-continuous. Let $\lambda_{\max}$ be the eigenvalue of ϕ of maximal modulus. We use the norm (2) on $\mathbb{R}^n$. Denote $B_r(x) = \{y \in \mathbb{R}^n :\ \|y - x\| < r\}$ and let $B_r := B_r(0)$.

Lemma 3.4. *The map f_γ is Hölder-continuous on $\mathcal{C}_\infty$: there exists $r > 0$ and $L_2 > 0$ such that for any $\xi_1, \xi_2 \in \mathcal{C}_\infty$, with $|\xi_1 - \xi_2| < r$ we have*

$$\|f_\gamma(\xi_1) - f_\gamma(\xi_2)\| \le L_2\|\xi_1 - \xi_2\|^\alpha, \quad \textit{for } \alpha = \frac{\log|\gamma|}{\log|\lambda_{\max}|}. \tag{12}$$

Proof. Let $r > 0$ be such that for every $x \in \mathbb{R}^n$ the ball $B_r(x)$ is covered by a tile containing x and its immediate neighbors; this is possible by FLC. Assume that $\delta = \|\xi_1 - \xi_2\| < r$ and $\xi_i = \phi^{-k}c_i$ for some $c_i \in \mathcal{C}$ and $k \in \mathbb{N}$. Define ℓ to be the smallest positive integer such that

$$\phi^k B_\delta(\phi^{-k}c_1) \subset \phi^\ell B_r(\phi^{-\ell}c_1).$$

Since $\ell \le k$, the last inclusion is equivalent to $|\lambda_{\max}|^{k-l}\delta \le r$, so we have

$$|\lambda_{\max}|^{-1}(r/\delta) \le |\lambda_{\max}|^{k-\ell} \le r/\delta. \tag{13}$$

Observe that

$$c_2 \in \phi^k\overline{B_\delta}(\phi^{-k}c_1) \subset \phi^\ell\overline{B_r}(\phi^{-\ell}c_1),$$

so $\phi^{-\ell}c_1$ and $\phi^{-\ell}c_2$ are in the same or in the neighboring tiles of $\mathcal{T}$ by the choice of r. We claim that there exists a finite set $W \subset J$, independent of c_1, c_2, such that

$$c_2 - c_1 = \sum_{i=0}^{\ell} \phi^i w_i \tag{14}$$

for some $w_i \in W$ (of course, w_i, as well as ℓ, depend on c_1, c_2). This is standard, but we provide a proof for completeness.

Let $T_i \in \mathcal{T}$ be such that $c_i = c(T_i)$, $i = 1, 2$. By the definition of SAT, there is a (unique) tile $T_i^{(1)} \in \mathcal{T}$ such that $\phi T_i^{(1)} \supset T_i^{(0)} := T_i$. Iterating this, we obtain a sequence of $\mathcal{T}$-tiles $T_i^{(j)}$, for $j \ge 0$, such that $\phi T_i^{(j)} \supset T_i^{(j-1)}$, for $j \ge 1$ and

$i = 1, 2$. Note that $T_i^{(\ell)} \supset \phi^{-\ell} T_i^{(0)} \ni \phi^{-\ell} c_i$, hence $T_1^{(\ell)}$ and $T_2^{(\ell)}$ either coincide or are adjacent. We have

$$\begin{aligned} c_2 - c_1 &= \sum_{j=0}^{\ell-1} \left[\left(\phi^j c(T_2^{(j)}) - \phi^{j+1} c(T_2^{(j+1)}) \right) - \left(\phi^j c(T_1^{(j)}) - \phi^{j+1} c(T_2^{(j+1)}) \right) \right] \\ &+ \phi^\ell c(T_2^{(\ell)}) - \phi^\ell c(T_1^{(\ell)}). \end{aligned}$$

This implies (14), since the set

$$\{c(T') - \phi c(T'') : \ T', T'' \in \mathcal{T}, \ T' \subset \phi T''\}$$

is finite by FLC, as well as the set Ψ from (3), to which w_ℓ belongs.

Now we can write, using (3), the additivity of the address map on J, and (8),

$$\begin{aligned} f_\gamma(c_1) - f_\gamma(c_2) &= \pi_\gamma a(c_2 - c_1) \\ &= \pi_\gamma a \left(\sum_{i=0}^{\ell} \phi^i w_i \right) \\ &= \sum_{i=0}^{\ell} M^i \pi_\gamma a(w_i). \end{aligned}$$

Thus, in view of (11) and (9),

$$\begin{aligned} \|f_\gamma(\phi^{-k} c_2) - f_\gamma(\phi^{-k} c_1)\| &= \|M^{-k}(f_\gamma(c_1) - f_\gamma(c_2))\| \\ &= |\gamma|^{-k} \|f_\gamma(c_1) - f_\gamma(c_2)\| \\ &= |\gamma|^{-k} \left\| \sum_{i=0}^{\ell} M^i \pi_\gamma a(w_i) \right\| \\ &\le |\gamma|^{-k} \sum_{i=0}^{\ell} |\gamma|^i \|\pi_\gamma a(w_i)\| \le L' |\gamma|^{\ell-k}, \end{aligned}$$

where $L' = \frac{|\gamma|}{|\gamma|-1} \max_{w \in W} \|a(w)\|$. In view of (13),

$$|\gamma|^{\ell-k} = (|\lambda_{\max}|^{\ell-k})^\alpha \le (|\lambda_{\max}|\delta/r)^\alpha = \text{const} \cdot \|\xi_1 - \xi_2\|^\alpha,$$

so we obtain the desired inequality. □

Now we extend f_γ by continuity and obtain a function $f_\gamma : \mathbb{R}^n \to U_\gamma$. Observe that

$$f_\gamma \circ \phi = M \circ f_\gamma, \tag{15}$$

since this holds on the dense set $\mathcal{C}_\infty$. We also have the following property.

Lemma 3.5. *Let E_θ be the real invariant subspace of ϕ corresponding to an eigenvalue θ and suppose that $|\gamma| \ge |\theta|$. Then $f_\gamma|_{E_\theta+x}$ is Lipschitz for any $x \in \mathbb{R}^n$, with a uniform constant $2L_1$ (where L_1 is the constant in Lemma 3.2). If $|\gamma| > |\theta|$, then $f_\gamma|_{E_\theta+x}$ is constant for any $x \in \mathbb{R}^n$.*

Proof. Let $\xi_1, \xi_2 \in \mathbb{R}^n$ be such that $\xi_2 - \xi_1 \in E_\theta$. By (15), we have for $k \in \mathbb{N}$,

$$\begin{aligned} \|f_\gamma(\xi_1) - f_\gamma(\xi_2)\| &= \|M^{-k}(f_\gamma(\phi^k\xi_1) - f_\gamma(\phi^k\xi_2))\| \\ &= |\gamma|^{-k}\|f_\gamma(\phi^k\xi_1) - f_\gamma(\phi^k\xi_2)\|. \end{aligned}$$

Let c_i be a nearest control point to $\phi^k\xi_i$; its distance to $\phi^k\xi_i$ is at most $d_{\max} = \max\{\mathrm{diam}(T) : T \in \mathcal{T}\}$. If k is so large that $\|\phi^k\xi_1 - \phi^k\xi_2\| > 2d_{\max}$, then $\|c_1 - c_2\| < 2\|\phi^k\xi_1 - \phi^k\xi_2\|$, and we have by uniform continuity of f_γ, Lemma 3.2, and (2), with a uniform constant C_3:

$$\begin{aligned} \|f_\gamma(\phi^k\xi_1) - f_\gamma(\phi^k\xi_2)\| &\le C_3 + \|f(c_1) - f(c_2)\| \\ &\le C_3 + L_1\|c_1 - c_2\| \\ &\le C_3 + 2L_1\|\phi^k\xi_1 - \phi^k\xi_2\| \\ &= C_3 + 2L_1|\theta|^k\|\xi_1 - \xi_2\|. \end{aligned}$$

Thus,

$$\|f_\gamma(\xi_1) - f_\gamma(\xi_2)\| \le C_3|\gamma|^{-k} + 2L_1(|\theta|/|\gamma|)^k\|\xi_1 - \xi_2\|.$$

The lemma follows by letting $k \to \infty$. (Recall that $|\gamma| \ge |\theta| > 1$.) □

Lemma 3.6. *The function f_γ depends only on the tile type in $\mathcal{T}$ up to an additive constant: if $T, T + x \in \mathcal{T}$ and $\xi \in T$, then*

$$f_\gamma(\xi + x) = f_\gamma(\xi) + \pi_\gamma a(x). \tag{16}$$

Observe that $x \in \mathcal{C} - \mathcal{C}$, so $a(x)$ is defined, but we cannot write $\pi_\gamma a(x) = f_\gamma(x)$, since we do not necessarily have $x \in \mathcal{C}$.

Proof. It is enough to check (16) on a dense set. Suppose $\xi = \phi^{-k}c(S) \in T$ for some $S \in \mathcal{T}$. Then $S \subset \phi^k T$ and $S + \phi^k x \subset \phi^k(T + x)$ so $S + \phi^k x \in \mathcal{T}$. Thus,

$$\begin{aligned} f_\gamma(\xi + x) &= f_\gamma(\phi^{-k}c(S) + x) \\ &= f_\gamma(\phi^{-k}c(S + \phi^k x)) \\ &= M^{-k} f_\gamma(c(S + \phi^k x)) \\ &= M^{-k} f_\gamma(c(S)) + M^{-k}\pi_\gamma a(\phi^k x) \\ &= f_\gamma(\xi) + \pi_\gamma a(x), \end{aligned}$$

as desired. Here we used the definition of f_γ on $\mathcal{C}$ and (8). □

Lemma 3.7. *If $|\gamma| \geq |\lambda|$ then $f_\gamma|_{E_\lambda + x}$ is a constant function for any $x \in \mathbb{R}^n$.*

Proof. By Lemma 3.5, this holds if $|\gamma| > |\lambda|$, so it remains to consider the case $|\gamma| = |\lambda|$. We know that for all $x \in \mathbb{R}^n$, the restriction $f_\gamma|_{E_\lambda + x}$ is Lipschitz, hence a.e. differentiable by Rademacher's Theorem. It follows that

$$D(x)u := \lim_{t \to 0} \frac{f_\gamma(x + tu) - f_\gamma(x)}{t}$$

exists for a.e. $x \in \mathbb{R}^n$ for all $u \in E_\lambda$, and is a linear transformation in u (from E_λ to U_γ). Moreover, $D(x)$ is measurable in x, since it is a limit of continuous functions. Since $D(x)$ is the total derivative, we have

(17)
$$\lim_{k \to \infty} \left(\sup_{u \in E_\lambda,\ 0 < \|u\| < 1/k} \frac{\|f_\gamma(x + u) - f_\gamma(x) - D(x)u\|}{\|u\|} \right) = 0 \quad \text{for a.e. } x \in \mathbb{R}^n.$$

The functions in parentheses are measurable and converge a.e., hence by Egorov's Theorem they converge uniformly on a set of positive measure. Uniform convergence means that there exists a sequence of positive integers $N_k \uparrow \infty$ such that

$$\begin{aligned} \Omega \;:=\; & \{\xi \in \mathbb{R}^n :\ \|f_\gamma(\xi + u) - f_\gamma(\xi) - D(\xi)u\| \leq \|u\|/k \\ & \forall\, u \in B_{1/N_k} \cap E_\lambda,\ \text{for all } k\} \end{aligned}$$

has positive Lebesgue measure. We claim that Ω has full Lebesgue measure.

Observe that if $T, T + x \in \mathcal{T}$ and $\xi \in T^\circ$, then

$$\xi \in \Omega \;\Rightarrow\; \xi + x \in \Omega \tag{18}$$

by Lemma 3.6. Furthermore, by (15) we have $D(\phi\xi) = MD(\xi)\phi^{-1}$ and, denoting $v = \phi u$, for all $v \in B_{|\lambda|/N_k} \cap E_\lambda$,

$$\begin{aligned} \|f_\gamma(\phi\xi + v) - f_\gamma(\phi\xi) - D(\phi\xi)v\| &= \|M(f_\gamma(\xi+u) - f_\gamma(\xi)) - D(\xi)u)\| \\ &= |\gamma| \cdot \|f_\gamma(\xi+u) - f_\gamma(\xi) - D(\xi)u\| \\ &\le |\gamma| \cdot \|u\|/k = |\lambda| \cdot \|u\|/k = \|v\|/k, \end{aligned}$$

where we used that $\phi|_{E_\lambda}$ expands the norm by a factor of $|\lambda|$. This shows that $\phi(\Omega) \subset \Omega$.

We will need a version of Lebesgue-Vitali Density Theorem where the differentiation basis is not the set of balls but rather the collection of sets of the form $\phi^{-k}B_1$, $k \ge 0$, and their translates. It is a well-known fact in Harmonic Analysis that such sets form a density basis, for any expanding linear map ϕ (even non-diagonalizable), see [23, pp. 8-13] or [17, pp. 11-14]. Let y be a density point of Ω, i.e., denoting the Lebesgue measure by m,

$$m(\Omega \cap \phi^{-k}B_1(\phi^k y)) \ge (1-\varepsilon_k)m(\phi^{-k}B_1) \quad \text{for some } \varepsilon_k \to 0.$$

Denote by $[B_1(x)]^{\mathcal{T}}$ the patch consisting of those tiles which intersect $B_1(x)$. By repetitivity, there exists $R > 0$ such that B_R contains a translate of $[B_1(x)]^{\mathcal{T}}$ for every $x \in \mathbb{R}^n$. Let $y_k \in B_R$ be such that $[B_1(y_k)]^{\mathcal{T}}$ is a translate of $[B_1(\phi^k y)]^{\mathcal{T}}$. Then

$$\begin{aligned} m(\Omega \cap B_1(y_k)) &= m(\Omega \cap B_1(\phi^k y)) \\ &\ge m(\phi^k\Omega \cap B_1(\phi^k y)) \\ &= |\det\phi|^k m(\Omega \cap \phi^{-k}B_1(\phi^k y)) \\ &\ge |\det\phi|^k(1-\varepsilon_k)m(\phi^{-k}B_1) = (1-\varepsilon_k)m(B_1). \end{aligned}$$

We used (18) and $\phi^k\Omega \subset \Omega$ in the first two displayed lines above. Let y' be a limit point of y_k. Then we have $m(\Omega \cap B_1(y')) = m(B_1)$. Thus, Ω is a set of full measure in $B_1(y')$, and by expansion and translation we conclude that Ω has full measure in $\mathbb{R}^n$, completing the proof of the claim.

Now choose ℓ_k so that $|\lambda|^{\ell_k} > N_k$. We have

$$\begin{aligned} \zeta \in \phi^{\ell_k}\Omega \quad \Rightarrow \quad & \|f_\gamma(\zeta+v) - f_\gamma(\zeta) - D(\zeta)v\| \le \|v\|/k \\ & \text{for all } v \in \phi^{\ell_k}(B_{1/N_k} \cap E_\lambda) \supset B_1 \cap E_\lambda. \end{aligned}$$

We know that $\Omega' = \bigcap_{k\ge 1} \phi^{\ell_k}\Omega$ has full measure, hence it is dense. For any $\xi \in \mathbb{R}^n$ choose a sequence $\xi_k \to \xi$ such that $D(\xi_k)$ converges (this is possible since $\|D(\xi)\| \le 2L_1$ by Lemma 3.5). Passing to the limit, we obtain that

$$f_\gamma(\xi + v) = f_\gamma(\xi) + D(\xi)v, \quad \text{for all } v \in B_1 \cap E_\lambda.$$

This shows that f is affine linear on every E_λ slice:

$$f_\gamma(\xi + v) = f_\gamma(\xi) + D(\xi)v, \quad \text{for all } v \in E_\lambda,$$

and $D(\xi) = D(\xi')$ whenever $\xi' - \xi \in E_\lambda$. Taking $\xi = 0$ we see that $f_\gamma|_{E_\lambda}$ is linear. It intertwines $\phi|_{E_\lambda}$ and $M|_{U_\gamma}$. But $\{\gamma, \overline{\gamma}\} \cap \{\lambda, \overline{\lambda}\} = \emptyset$ which are the eigenvalues of $\phi|_{E_\lambda}$ and $M|_{U_\gamma}$ respectively, hence the only possibility is $f_\gamma|_{E_\lambda} \equiv 0$. Since f_γ is uniformly continuous on $\mathbb{R}^n$ and $f_\gamma|_{x+E_\lambda}$ is affine linear, we obtain that $f_\gamma|_{x+E_\lambda} \equiv \text{const}(x)$. □

To motivate the conclusion of the proof, we start with a heuristic discussion. Assume that $|\gamma| \ge |\lambda|$ for the rest of the proof. So far, we have proved that f_γ is affine linear on the slices $x + E_\lambda$. Suppose we could show that f_γ is linear on $\mathbb{R}^n$. Then we could conclude as follows: $f_\gamma \circ \phi = M \circ f_\gamma$ and $f_\gamma(\mathbb{R}^n) = U_\gamma$ (the latter follows from (5) and the definition of f_γ) would imply that ϕ restricted to a linear subspace and $M|_{U_\gamma}$ are linearly conjugate:

$$\begin{array}{ccc} U_\gamma \subset \mathbb{R}^N & \xrightarrow{M} & U_\gamma \subset \mathbb{R}^N \\ f_\gamma \big\uparrow & & f_\gamma \big\uparrow \\ \mathbb{R}^n & \xrightarrow{\phi} & \mathbb{R}^n \end{array}$$

and hence γ is an eigenvalue of ϕ of multiplicity at least $\dim U_\gamma \ge \dim E_\lambda$, as desired.

This scheme does work, but with some modifications. We are able to show that f_γ is affine linear in some, but possibly not all, directions complementary to E_λ. It is linear in directions for which the differences between control points for tiles of the same type project densely.

Let $\Xi = \Xi(\mathcal{T})$ denote the set of translation vectors between tiles of the same type and let P_λ be the projection from $\mathbb{R}^n$ to E_λ commuting with ϕ (note that the projection π_γ acts in another space, $\mathbb{R}^N$).

Consider the set $(I-P_\lambda)\Xi$, that is, the projection of Ξ onto the other eigenspaces of ϕ. This projection may look like a lattice in some directions and fail to be discrete in other directions. We consider the directions in which this set is not discrete; more precisely, those directions in which there are arbitrarily small nonzero vectors in $(I-P_\lambda)\Xi$, and denote the span of these directions E'. What we will prove is that f_γ is affine linear on all E' slices, and hence all $E' \oplus E_\lambda$ slices. We will then show that the subspace $E := E' \oplus E_\lambda$ is ϕ-invariant and is spanned by the vectors of Ξ contained in it. This will allow us to essentially restrict the entire construction to $\phi|_E$ and conclude as indicated above, using that $f_\gamma|_E$ is linear.

Now let us be more formal and for each $\varepsilon > 0$ define $E_\varepsilon \subset \mathbb{R}^n$ to be the subspace

$$E_\varepsilon = \mathrm{Span}_{\mathbb{R}}(B_\varepsilon \cap (I-P_\lambda)\Xi) \subset E_\lambda^\perp \subset \mathbb{R}^n,$$

where $E_\lambda^\perp$ is the ϕ-invariant subspace complementary to E_λ. Further, consider

$$E' := \bigcap_{\varepsilon>0} E_\varepsilon.$$

We have $\phi\Xi \subset \Xi$ and $P_\lambda\phi = \phi P_\lambda$, hence

$$\phi((I-P_\lambda)\Xi) \subset (I-P_\lambda)\Xi.$$

Note that E_ε are decreasing linear subspaces of $E_\lambda^\perp \subset \mathbb{R}^n$, hence $E' = E_\varepsilon$ for some $\varepsilon > 0$, and so $E' = E_{\varepsilon'}$ for all $0 < \varepsilon' \le \varepsilon$. Since $\phi E_{\varepsilon'} \subset E_{c\varepsilon'}$ for $c = \|\phi\|$ we see that E' is ϕ-invariant. We then define

$$E := E' + E_\lambda.$$

Lemma 3.8. *$f_\gamma|_{E+x}$ is affine linear for every $x \in \mathbb{R}^n$.*

Proof. Choose ε so that $E' = E_\varepsilon$. Let $\varepsilon' < \varepsilon$ and define

$$E'' := \mathrm{Span}(B_{\varepsilon'} \cap (I-P_\lambda)(\mathcal{C}_1 - \mathcal{C}_1))$$

where $\mathcal{C}_1$ is the set of control points of tiles of type 1 (of course, we could equally well choose another tile type). First we claim that

$$E' = E''. \tag{19}$$

Indeed, $\mathcal{C}_1 - \mathcal{C}_1 \subset \Xi$ hence $E'' \subset E'$. Choose ℓ so large that $\phi^\ell\Xi \subset \mathcal{C}_1 - \mathcal{C}_1$; such an ℓ exists by primitivity of the tile substitution (the ℓ-th power of the substitution of any tile contains tiles of all types). We then have

$$E' = \phi^\ell E' = \phi^\ell E_{\varepsilon'/\|\phi\|^\ell} \subset \mathrm{Span}(B_{\varepsilon'} \cap (I-P_\lambda)\phi^\ell\Xi) \subset E''.$$

The claim is proved.

Now suppose $x \in \mathcal{C}_1 - \mathcal{C}_1$, so there exists $T \in \mathcal{T}$ of type 1 such that $T + x \in \mathcal{T}$. By Lemma 3.6,

$$\xi \in T \ \Rightarrow\ f_\gamma(\xi + x) = f_\gamma(\xi) + \pi_\gamma a(x).$$

But Lemma 3.7 implies that $f_\gamma(\xi + x) = f_\gamma(\xi + x - P_\lambda x)$, so

$$f_\gamma(\xi + (I - P_\lambda)x) = f_\gamma(\xi) + \pi_\gamma a(x) \quad \text{for } \xi \in T. \tag{20}$$

We want to show that f_γ is affine linear on all E-slices. Since f_γ is constant on all E_λ-slices by Lemma 3.7, it is enough to verify that f_γ is affine linear on all E'-slices (recall that $E = E' + E_\lambda$). Fix a small ε' as in (19) and select a basis of E' of the form $y_i = (I - P_\lambda)x_i \in B_{\varepsilon'}$, with $x_i \in \mathcal{C}_1 - \mathcal{C}_1$, for $i = 1, \dots, \dim E'$. Now for any ξ in the interior of T, such that $B_r(\xi) \subset T$, we obtain from (20):

$$f_\gamma\Big(\xi + \sum_i b_i y_i\Big) = f_\gamma(\xi) + \sum_i b_i \pi_\gamma a(x_i),$$

for all $b_i \in \mathbb{Z}$ such that $\sum_i b_i y_i \in B_r$. (Here we should note that, in view of Lemma 3.6, equality (20) transfers to all tiles equivalent to T. Since all the x_i are translates between two copies of T, we can apply the equality for any x_i in any of the translates.) This shows that f_γ is affine linear on a large chunk of the lattice in E' generated by small vectors y_i, translated in such a way that ξ becomes the origin. It is an easy exercise to pass to the limit as $\varepsilon' \to 0$ and conclude that f_γ is affine linear in the E'-direction on $B_r(\xi) \cap (E' + \xi)$. To be a bit more precise, we can verify that

$$f_\gamma\Big(\frac{\zeta_1 + \zeta_2}{2}\Big) = \frac{f_\gamma(\zeta_1) + f_\gamma(\zeta_2)}{2} \quad \text{for all } \zeta_1, \zeta_2 \in B_r(\xi) \cap (E' + \xi). \tag{21}$$

Since f_γ is continuous, this implies that

$$f_\gamma(\zeta) = A_\xi \zeta + b_\xi \quad \text{for all } \zeta \in B_r(\xi) \cap (E' + \xi), \tag{22}$$

see e.g., [1, 2.1.4], where it is called the "Jensen functional equation". The details are straightforward.

Since (22) holds on all slices of T, by "expanding and translating" with the help of (15) and Lemma 3.6, we obtain the claim of the lemma. □

Lemma 3.9.

$$E = \mathrm{Span}_{\mathbb{R}}((\mathcal{C} - \mathcal{C}) \cap E).$$

Proof. Denote $W := \mathrm{Span}_{\mathbb{R}}((\mathcal{C}-\mathcal{C})\cap E)$. First we show that $E_\lambda \subset W$. Let $w \in E_\lambda$. The set $\mathcal{C}_1$ (control points of type-1 tiles) is relatively dense in $\mathbb{R}^n$; let $R > 0$ be such that every open ball of radius R hits $\mathcal{C}_1$. Let $\xi_j \in \mathcal{C}_1$ be such that $\|\xi_j - jw\| < R$ for all $j \geq 0$. Then

$$\|(I-P_\lambda)\xi_j\| = \|(I-P_\lambda)(\xi_j - jw)\| \leq (1+\|P_\lambda\|)R, \quad j \geq 0.$$

It follows that there exists a sequence of pairs (i_k, j_k), with $i_k - j_k \to +\infty$, such that

$$\|(I-P_\lambda)(\xi_{i_k} - \xi_{j_k})\| \to 0, \quad \text{as } k \to \infty.$$

Therefore, $(I-P_\lambda)(\xi_{i_k} - \xi_{j_k}) \in E'$ for k sufficiently large, and hence $\xi_{i_k} - \xi_{j_k} \in E$ for $k \geq k_0$. Now,

$$\|(\xi_{i_k} - \xi_{j_k}) - w(i_k - j_k)\| \leq 2R,$$

hence $\zeta_k := (\xi_{i_k} - \xi_{j_k})/(i_k - j_k) \to w$. But $\zeta_k \in W$, for $k \geq k_0$, hence $w \in W$, since W is closed, being a linear subspace of $\mathbb{R}^n$.

Now recall that E' is spanned by certain vectors of the form $\xi - P_\lambda \xi$, with $\xi \in \Xi \subset \mathcal{C} - \mathcal{C}$. Since $P_\lambda \xi \in E_\lambda \subset E$, we have that these vectors ξ are in E, and hence $E' \subset W$. This proves that $E = E' + E_\lambda \subset W$, as desired. □

Conclusion of the proof of Theorem 3.1. As mentioned earlier, we would like to run the entire construction essentially restricting ourselves to the subspace E, which is ϕ-invariant, contains E_λ, and is spanned by the vectors of $\mathcal{C}-\mathcal{C}$ in it. We do not literally do this, because it is not clear what the intersection of the tiling with E looks like; rather, we make sure that the construction on $\mathbb{R}^n$ is compatible with this subspace structure. Recall that at the beginning of the proof we considered the free Abelian group $J = \langle \mathcal{C} \rangle$ and its free generators $v_1, \ldots, v_N$. We will now use a more specific choice of the generators. Namely, let

$$\widetilde{J} := \langle (\mathcal{C}-\mathcal{C})\cap E\rangle = \mathrm{Span}_{\mathbb{Z}}((\mathcal{C}-\mathcal{C})\cap E).$$

Clearly, $\widetilde{J}$ is an Abelian subgroup of J, and $\mathrm{Span}_{\mathbb{R}}\widetilde{J} = E$ by Lemma 3.9. Is it possible to choose the free generators for J as an extension of a set of free generators for $\widetilde{J}$? Maybe not, but we can choose $v_1, \ldots, v_N$, the free generators of J, so that $d_1 v_1, \ldots, d_s v_s$ are free generators of $\widetilde{J}$ for some positive integers d_j and $s \leq N$ (see e.g. [8, Theorem II.1.6]).

Recall that ϕ acts on J, and on the generators v_j this action is given by an integer matrix M. Since ϕ also acts on $\widetilde{J}$, we claim that $M = \left(\begin{array}{c|c} \widetilde{M} & * \\ \hline 0 & * \end{array}\right)$, where $\widetilde{M}$ is an $s \times s$ matrix. Indeed, $\phi(v_i)$, $i \le N$, is a unique integral linear combination of $\{v_j\}_{j \le N}$, with the coefficients coming from the i-th column of M. On the other hand, $\phi(d_i v_i)$, $i \le s$, is an integral linear combination of $\{d_j v_j\}_{j \le s}$, since the latter are free generators of $\widetilde{J}$. This implies that $\phi(v_i)$, $i \le s$, is an integral linear combination of $\{d_j v_j\}_{j \le s}$, that is,

$$\phi[v_1 \dots v_s] = [v_1 \dots v_s]\widetilde{M}, \tag{23}$$

where $\widetilde{M}$ is an integral $s \times s$ matrix. Thus, the matrix M is block upper-triangular, with the upper left corner $\widetilde{M}$, as claimed above.

Note that

$$\mathrm{Span}_{\mathbb{R}}(\{v_j\}_{j \le s}) = \mathrm{Span}_{\mathbb{R}}(\{d_j v_j\}_{j \le s}) = \mathrm{Span}_{\mathbb{R}}((\mathcal{C} - \mathcal{C}) \cap E) = E \tag{24}$$

by construction. By (23) and (24), there is an $\widetilde{M}$-invariant subspace of $\mathbb{R}^s$, on which $\widetilde{M}$ acts isomorphically (linearly conjugate) to $\phi|_E$. Since $E \supset E_\lambda$, we obtain that λ is an eigenvalue of $\widetilde{M}$, with the multiplicity greater or equal to $\dim E_\lambda$. Because γ is an algebraic conjugate of λ and $\widetilde{M}$ is an integer matrix, we have that γ is also an eigenvalue of $\widetilde{M}$, with the multiplicity $\ge \dim E_\lambda$. Let $\widetilde{U}_\gamma$ be the real invariant subspace of $\widetilde{M}$ corresponding to γ.

Abusing notation a bit, we will identify $\mathbb{R}^s$ with the subspace of $\mathbb{R}^N$ generated by the first s coordinates. Then $\widetilde{U}_\gamma \subset U_\gamma$.

Let $a : J \to \mathbb{Z}^N$ be the address map, as in (4). Then $a(\widetilde{J}) \subset \mathbb{Z}^s$ (using a similar abuse of notation, so that $\mathbb{Z}^s \subset \mathbb{Z}^N$). By construction,

$$\mathrm{Span}_{\mathbb{Z}}\{a(\xi - \xi') :\ \xi, \xi' \in \mathcal{C},\ \xi - \xi' \in E\} = \bigoplus_{j=1}^{s} d_j \mathbb{Z} \subset \mathbb{Z}^s,$$

hence

$$\mathrm{Span}_{\mathbb{R}}\{a(\xi - \xi') :\ \xi, \xi' \in \mathcal{C},\ \xi - \xi' \in E\} = \mathbb{R}^s.$$

It follows that

$$\mathrm{Span}_{\mathbb{R}}\{\pi_\gamma(a(\xi) - a(\xi')) :\ \xi, \xi' \in \mathcal{C},\ \xi - \xi' \in E\} = \pi_\gamma(\mathbb{R}^s) = \widetilde{U}_\gamma. \tag{25}$$

Recall that $f_\gamma : \mathbb{R}^n \to \mathbb{R}^N$, defined originally by $f_\gamma(\xi) = \pi_\gamma(a(\xi))$ on control points, is uniformly continuous, $f_\gamma \circ \phi = M \circ f_\gamma$, and $f_\gamma|_{E+x}$ is affine linear for all x by Lemma 3.8. Note that $f_\gamma|_E$ is linear, since $f_\gamma(0) = 0$.

We claim that $f_\gamma(E) \supset \widetilde{U}_\gamma$. Indeed, every $f_\gamma(E+x)$ is a translate of a linear subspace, which must be a translate of $f_\gamma(E)$, by the uniform continuity of f_γ. It follows that for $\xi, \xi' \in \mathcal{C},\ \xi - \xi' \in E$,

$$\pi_\gamma(a(\xi) - a(\xi')) = f_\gamma(\xi) - f_\gamma(\xi') \in f_\gamma(E),$$

whence $\widetilde{U}_\gamma \subset f_\gamma(E)$ by (25). The claim is verified.

Since $f_\gamma(E)$ contains $\widetilde{U}_\gamma$, there exists a ϕ-invariant subspace $\widetilde{E} \subset E \subset \mathbb{R}^n$, such that f_γ maps $\widetilde{E}$ isomorphically onto $\widetilde{U}_\gamma$:

$$\begin{array}{ccc} \widetilde{U}_\gamma \subset \mathbb{R}^s & \xrightarrow{\widetilde{M}} & \widetilde{U}_\gamma \subset \mathbb{R}^s \\ f_\gamma \Big\uparrow & & f_\gamma \Big\uparrow \\ \widetilde{E} \subset E & \xrightarrow{\phi} & \widetilde{E} \subset E \end{array}$$

Thus, the linear map $f_\gamma|_{\widetilde{E}}$ conjugates $\phi|_{\widetilde{E}}$ to $\widetilde{M}|_{\widetilde{U}_\gamma} = M|_{\widetilde{U}_\gamma}$, hence γ is an eigenvalue of ϕ of multiplicity $\geq \dim E_\lambda$, as desired. $\square$

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